Network Flow Notes

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Definition 1. *A network is a directed graph where:*

- *Each edge e is assigned a positive real number $C_e > 0$, called its* capacity*.*
- *There is a non-empty set of vertices S, where each vertex in S is called a* source*.*
- *There is a non-empty set of vertices T called* sinks*.*
- *We require that S and T are disjoint.*

Definition 2. *Let G be a network. A* flow *f on G is an assignment of a real number f_e to each edge e in the network. We require that:*

- *For all edges $0 \leq f_e \leq C_e$.*
- *(Consevation of flow property) For all vertices v not a source or a sink*

$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

Definition 3. *The* size *of a flow f, denoted $|f|$ is defined by*

$$|f| = \sum_{e \text{ out of some source}} f_e - \sum_{e \text{ into some source}} f_e$$

Lemma 1. *Let f be a flow on a network $G = (V, E)$ with sources S and sinks T. Then*

$$|f| = \sum_{e \text{ into some sink}} f_e - \sum_{e \text{ out of some sink}} f_e$$

Proof. We observe that

$$\begin{aligned}
0 &= \sum_{e\in E} f_e - \sum_{e\in E} f_e \\
&= \sum_{v\in V}\left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e\right)
\end{aligned}$$

if v is neither a source nor a sink then $\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e = 0$ by conservation of flow,

$$= \sum_{v\in S\cup T}\left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e\right)$$

since S and T are disjoint we can split the sum

$$\begin{aligned}
&= \sum_{v\in S}\left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e\right) + \sum_{v\in T}\left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e\right) \\
&= \sum_{v\in S}\sum_{e \text{ into } v} f_e - \sum_{v\in S}\sum_{e \text{ out of } v} f_e \\
&\qquad + \sum_{v\in T}\sum_{e \text{ into } v} f_e - \sum_{v\in T}\sum_{e \text{ out of } v} f_e \\
&= \sum_{e \text{ into some source}} f_e - \sum_{e \text{ out of some source}} f_e \\
&\qquad + \sum_{e \text{ into some sink}} f_e - \sum_{e \text{ out of some sink}} f_e \\
&= -|f| + \sum_{e \text{ into some sink}} f_e - \sum_{e \text{ out of some sink}} f_e.
\end{aligned}$$

So

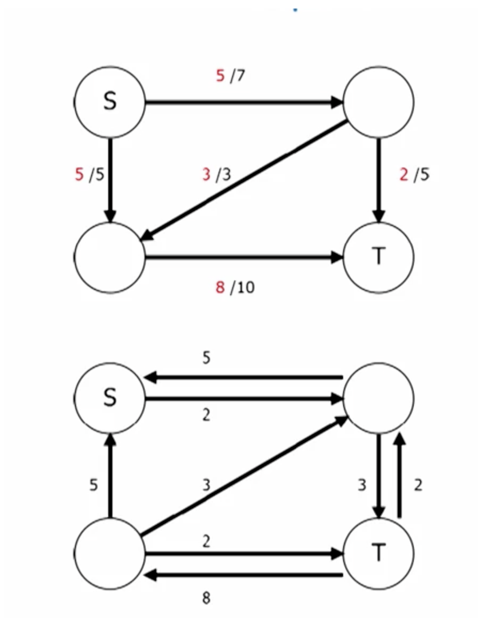
$$|f| = \sum_{e \text{ into some sink}} f_e - \sum_{e \text{ out of some sink}} f_e$$

as required. □

Definition 4. *Let G be a network. Let f be a flow on G. The* residual network *G_f is a graph with the same vertex set as G with the following edges:*

- *For every edge e of G with $f_e < C_e$ there is an edge in G_f with same beginning and end vertex and with capacity $C_e - f_e > 0$.*
- *For every edge e of G with $f_e > 0$ there is an edge e' in the opposite direction in G_f with capacity f_e.*

Example 1. The image at the top is a graph with edges labeled f_e/C_e. The bottom image is the residual graph is the residual graph where each edge is labelled with its capacity.

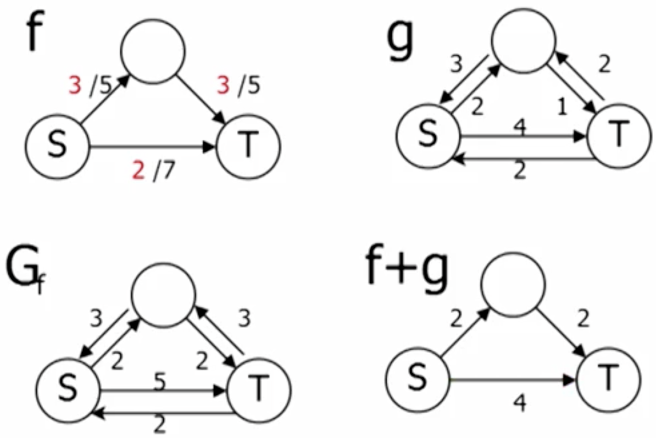


Definition 5. *Let f be a flow on a network G. Let g be a flow on the residual G_f. Then we define a flow $f + g$ on G as follows:*

- *Let e be an edge in G with flow f_e.*
- *Let g_e be the flow along the corresponding edge in G_f.*
- *Let e' be the edge in G_f in the opposite direction to e. It has flow g'_e.*
- *Then $(f + g)_e = f_e + g_e - g_{e'}$.*

(The next theorem proves that $f + g$ is indeed a flow).

Example 2. An example of the sum of two flows is given in this diagram:



Theorem 1. *Let G be a network with flow f. Let G_f be its residual with flow g. Then:*

- *$f + g$ is a flow on G*
- $|f + g| = |f| + |g|$
- *Fix f. Then for any flow f' on G there exists some g' such that $f' = f + g'$.*

Proof. Conservation of flow of f and g imply conservation of flow of $f + g$. Also $f_e + g_e \leq f_e + (c_e - f_e) = C_e$ and $f_e - g_{e'} \geq f_e - f_e = 0$ thus $0 \leq f_e + g_e - g_{e'} \leq C_e$ so $(f + g)_e$ satisfies the conditions in the definition of flow for all edges e.
For any particular source s we have

$$\begin{aligned}\sum_{e \text{ out of } s} (f+g)_e - \sum_{e \text{ into } s} (f+g)_e &= \sum_{e \text{ out of } s} (f_e + g_e - g_{e'}) - \sum_{e \text{ into } s} (f_e + g_e - g_{e'}) \\ &= \sum_{e \text{ out of } s} f_e - \sum_{e \text{ into } s} f_e + \sum_{e \text{ out of } s} (g_e - g_{e'}) - \sum_{e \text{ into } s} (g_e - g_{e'}).\end{aligned}$$

(Note the sums are over edges in G not G_f.) If we take the first and last expression in the displayed equation and sum over all sources s we get $|f + g| = |f| + |g|$.

(Details omitted) For the last part we can define $g' = f' - f$ for a natural definition of subtracting flows (flip the signs in the definition of the sum of two flows). This g' is the one we want. $\square$

Definition 6. *Given a network G a cut $\mathcal{C}$ is a set of vertices in G so that the set of sources S is a subset of $\mathcal{C}$ and $\mathcal{C}$ is disjoint from the sinks of G. The size of a cut is given by*

$$|\mathcal{C}| = \sum_{e \textit{ out of } \mathcal{C}} C_e$$

Lemma 2. *Fix a network. For any flow f and any cut $\mathcal{C}$ we have*

$$|f| \leq |\mathcal{C}|.$$

Proof. We have

$$|f| = \sum_{v \text{ source}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right)$$

now we use conservation of flow to add terms to the sum

$$= \sum_{v \in \mathcal{C}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right)$$

if both ends of an edge e are in $\mathcal{C}$ then these terms will cancel out of the sum

$$\begin{aligned} &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\ &\leq \sum_{e \text{ out of } \mathcal{C}} f_e \\ &\leq \sum_{e \text{ out of } \mathcal{C}} C_e \\ &= |\mathcal{C}|. \end{aligned}$$

$\square$

Corollary 1. *For any network*

$$\max_{\textit{flows } f} |f| \leq \min_{\textit{cuts } \mathcal{C}} |\mathcal{C}|.$$

We now strengthen this corollary and turn it into a theorem.

Theorem 2. *For any network*

$$\max_{\textit{flows } f} |f| = \min_{\textit{cuts } \mathcal{C}} |\mathcal{C}|.$$

Proof. If $\max_{\text{flows } f} |f| = 0$ then there is no path from any source to any sink. Let $\mathcal{C}$ be the set of vertices reachable from S the set of all sources. There are no edges out of $\mathcal{C}$ so $|\mathcal{C}| = 0$. □